Noise-enhanced measurement of weak doublet spectra with a Fourier-transform spectrometer and a 1-bit analog-to-digital converter

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We demonstrate an efficient noise dithering procedure for measuring the power spectrum of a weak spectral doublet with a Fourier-transform spectrometer in which the subthreshold interferogram is measured by a 1-bit analog-to-digital converter without oversampling. In the absence of noise, no information is obtained regarding the doublet spectrum because the modulation term $s(x)$ of its interferogram is below the instrumental detection limit $B$, i.e., $|s(x)| < B$, for all path difference $\Delta x$ values. Extensive numerical experiments are carried out concerning the recovery of the doublet power spectrum that is represented by $s(x) = (s_0/2)\exp(-\pi \Delta x^2/[\beta(\cos(2\pi f_1 x) + \cos(2\pi f_2 x)])$, where $s_0$ is a constant, $\beta$ is the linewidth factor, and $\langle f \rangle = (f_1 + f_2)/2$. Different values of $\langle f \rangle$, $s_0$, and $\beta$ are considered to evaluate thoroughly the accuracy of the procedure to determine the unknown values of $f_1$ and $f_2$, the spectral linewidth, and the peak values of the spectral profiles. Our experiments show that, even for short observation times, the resonant frequencies of $s(x)$ could be located with high accuracy over a wide range of $\langle f \rangle$ and $\beta$ values. Signal-to-noise ratios as high as 50 are also gained for the recovered power spectra. The performance of the procedure is also analyzed with respect to another method that recovers the amplitude values of $s(x)$ directly. © 2001 Optical Society of America

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1. Introduction

In recent years noise-enhanced detection of weak physical disturbances has attracted considerable attention particularly in the area of stochastic resonance (SR), which is the phenomenon that happens in nonlinear systems in which a weak signal is magnified and optimized by the addition of noise.\textsuperscript{1–3} That noise could improve (rather than degrade) the signal-to-noise ratio (SNR) of a measurement that was used more than 45 years ago\textsuperscript{4} in a technique called dithering in which the quantization errors of a low-bit-count analog-to-digital converter (ADC) are randomized by the addition of an uncorrelated signal to the weak signal $s(x)$ prior to its detection.

In SR, $s(x)$ is dithered with noise $n_{\sigma}(x)$, where $\sigma^2$ is the noise variance and $\langle n_{\sigma}(x) \rangle = 0$, where $\langle \rangle$ indicates the average value. As a signal recovery tool, SR is most effective when $s(x)$ is a subthreshold sinusoid of frequency $f_0$. The $\sigma$ value is chosen to optimize the SNR value of the $f_0$ line in the power spectrum $P_\sigma(f)$ of the dithered signal $s(x) + n_\sigma(x)$, where $f \geq 0$. SR has been observed in a wide variety of physical systems such as the Schmitt trigger, tunnel diode, bistable ring laser, and mechanoreceptor neurons in crayfish and crickets.\textsuperscript{1–3} Recently it was found that the paddle fish utilizes SR to enhance its ability to detect the electrical signals that are generated from its planktonic prey.\textsuperscript{5}

Noise dithering that is due to its optimal magnification of weak oscillations is a powerful (and counterintuitive) concept for devising new optical measurement techniques, but few have been reported so far. The cooperative effect of noise has been utilized in the nonlinear transmission of spike signals\textsuperscript{6} as well as weak spatial images through a Raman medium.\textsuperscript{7} Noise dithering has also been employed to measure the weak (square-wave) transmittance signal from a strongly absorbing medium\textsuperscript{8} in which the amplitude of the first harmonic of the transmittance signal was determined.

In this paper we demonstrate a new dithering technique for determining accurately the power spectrum...
of a weak spectral doublet using a Fourier-transform (FT) spectrometer that utilizes a 1-bit ADC. A spectral doublet is considered weak if the modulation (ac) term \( s(x) \) of its corresponding interferogram is below the instrumental detection limit \( B \) and therefore no information about the light source spectrum can be obtained. Parameter \( x \) is the optical path difference of the two beams in the Michelson interferometer. Spectral doublets are important in the assessment of information about the light source spectrum can be obtained. Previous SR-based spectral recovery procedures are largely ineffective for the task at hand, and the formulation of a new spectral recovery procedure is essential.

Our proposed spectral recovery procedure is developed within the framework of the sinusoid-crossing (SC) sampling technique.\(^{12-14}\) It makes use of the fact that the power spectrum of \( s(x) \) can be approximated accurately without the actual derivation of \( s(x) \) itself. By use of the ideal interpolation formula,\(^{15}\) the procedure is more sensitive than the one previously developed for the recovery of weak sinusoids that employed the cubic-spline interpolation technique.\(^{11}\)

In practice, the spectral recovery procedure can be implemented with a FT spectrometer that utilizes a 1-bit ADC to measure above-threshold interferograms without oversampling.\(^{16}\) To our knowledge, our research represents the first workable proposal regarding noise-enhanced measurement of subthreshold doublet spectra with such a FT spectrometer. Note that a 1-bit ADC is the simplest one possible and is a highly nonlinear device.

Single-bit coding continues to attract the attention of scientists and instrumentation engineers. Cross-sampling techniques (e.g., sigma-delta modulation sampling\(^{17-20}\) and SC sampling) are examples of single-bit coding that is implemented with a 1-bit ADC. High-accuracy conversion without oversampling is a distinct advantage of SC sampling compared with sigma-delta sampling.\(^{16}\)

For above-threshold input signals, single-bit coding has been utilized already in FT spectrometry\(^{16-21}\) and other applications.\(^{23-28}\) In a 1-bit ADC, analog conversion is implemented with only one comparator, whereas a Q-bit (flash) ADC (\( Q > 1 \)) requires \( Q \) comparators. Fewer comparators can be used in a (ladder-type) Q-bit ADC but at the expense of conversion speed. Because of its lower component count, a 1-bit ADC is easier to design, consumes less power, suffers less electronic noise, and converts at faster sampling rates.

Our presentation proceeds as follows: The principles of the SC sampling technique and noise dithering in SC sampling are discussed briefly in Subsections 2.A and 2.B, respectively. In Subsection 2.C we describe a noise dithering procedure for determining directly the unknown amplitude values of \( s(x) \). The power spectrum can be calculated from the recovered amplitude values. However, the said approach is inefficient because it requires long observation times for accuracy. The new spectral recovery technique is then described in Subsection 2.D. Unlike the previous approach, it achieves the same degree of accuracy at a considerably shorter observation time and is therefore applicable to short-duration signals.

We thoroughly evaluate the efficacy of the recovery procedure by considering different values of the doublet resonances and spectral linewidths. The quality of the recovered \( P_s(f) \) is evaluated relative to the true doublet power spectrum by use of Linfoot’s criteria of fidelity, structural content, and correlation quality.\(^{29,30}\) The results of our numerical experiments are presented and discussed in Sections 3 and 4, respectively. Appendix A lists the symbols of the various physical parameters used in the present paper.

2. Theory

A. Sinusoid-Crossing Sampling and Multithreshold Sampling

Real interferometers have finite detection limits that in many practical cases are due to the quantization error that is present in any analog-to-digital conversion process that utilizes a finite number of bits. ADC’s are essential in a FT spectrometer that relies on postdetection data processing to determine the corresponding emission (absorption) spectrum of the light source (sample).

For a Q-bit ADC with a dynamic range of \( 2A [-A \leq s(x) \leq A] \), the smallest detectable change in signal amplitude is \( 2A/(2^Q - 1) \), which means that \( s(x) \) values that are within the range \(-A/(2^Q - 1) < s(x) < A/(2^Q - 1) \) are indistinguishable from each other within the sampling period \( T \). In practice we deal with the equally sampled amplitude representation \( s(m) \) of \( s(x) \) \( w(x) \), where index \( m = 1, 2, \ldots, 2M = T/\Delta \), and \( \Delta \) is the sampling period. The sampling window function \( w(x) \) is defined as \( w(x) = 1 \) for \( 0 \leq x \leq T \), and \( w(x) = 0 \) for other \( x \) values.Erroneously, \( s(m) \) is a sequence of zero-amplitude values for any subthreshold \( s(x) \).

SC sampling can be implemented with a 1-bit (\( Q = 1 \)) ADC because the objective is to determine only the locations \( \{x_m\} \) where \( s(x) \) intersects with the reference sinusoid \( r(x) = A_r \cos(2\pi f_r x) \), where \( f_r = 1/2f_m \) and \( x_m \) is the SC location in the \( m \)th interval \( \Delta_m \).\(^{12-14}\) Each \( x_m \) is a solution to \( s(x) - r(x) = 0 \), and \( \{r(x_m)\} \) is an unequally sampled representation of \( s(x) \). The discrete Fourier spectrum \( \{S(k)\} \) of \( s(x) \) cannot be calculated accurately from \( \{r(x_m)\} \) because the latter is unequally sampled, where \( f = k/T \) and \( k = -M, -M + 1, \ldots, M \).\(^{15}\) A 1-bit ADC is essentially a threshold detector that could be developed from one comparator.

Each \( \Delta \) is measured between two successive ex-
extrema of $r(x)$ and hence all $\Delta_m$’s are centered at locations where $r(x) = 0$. SC sampling satisfies the Nyquist sampling criterion, and a SC exists within each $\Delta$ if (1) $f_r \geq f_c$ is the highest frequency component of $s(x)$ and (2) $A_r \geq |s(x)|$ for all $x$ within $T$. The corresponding analog signal $s(x)$ can be derived from $\{x_m\}$ either through Newton’s formula or through the ideal interpolation formula.15

In practice, $x_m$ is determined when we divide $\Delta_m$ into $N$ partitions of size $\delta$ (i.e., $\Delta = N\delta$), and its value is given by $x_m = (m - 1)\Delta + n\delta = [(m - 1)N + n]\delta$, where index $n = 0, 1, \ldots, N - 1$. It is determined with infinite accuracy within $\Delta_m$ only when $N$ is infinitely large ($\delta = 0$). With real SC circuits, $\delta$ cannot be made smaller than the circuit response time, and therefore the detection limit $B = \pi A_r/2N$ is always finite ($B > 0$). Note that for a given $r(x)$, the detection limit $B$ improves with increasing $N$. The center $x_{cm}$ of $\Delta_m$ is at $x_{cm} = (m - 1)\Delta + N\delta/2$, which implies that $\{s(m)\} = \{s(x_{cm})\}$. The SC sampling of $s(x)$ can be interpreted as multithreshold sampling that involves $N$ levels with the amplitude value of each level given by $r_n = A_r\cos(\pi n/N)$. The amplitude separation $\Delta r_n$ between two successive levels is $\Delta r_n = (\pi A_r/N)\sin(\pi n/N)$. Note that the separations are not equally spaced for all $n$. The largest separation occurs at the center of $\Delta$ and is given by $[\Delta r_{0.5N}] = r_{0.5N+1} = 0.5N = r_{0.5N} - r_{0.5N-1} = B$, whereas the smallest separation is between levels $x = (m - 1)\Delta$ and $x = (m - 1)\Delta \pm \delta$, which are located at the extrema of $r(x)$.

A distinct advantage of SC sampling is that it implements multithreshold sampling with a 1-bit ADC. In contrast, a standard 2N-level (fixed-level) multithreshold sampling requires 2N comparators. Note also that sampling with a bistable dc crossing detector that locates the crossings of $y(x)$ with respect to $\pm B$ levels does not satisfy the Nyquist criterion.

The power spectra of above-threshold interferograms have been determined previously with a FT spectrometer that utilizes SC sampling with a 1-bit ADC.16

B. Noise Dithering in Sinusoid-Crossing Sampling

In the present case, $s(x)$ is considered undetectable by SC sampling because $-B < s(x) < B$. Hence the SC’s of $s(x)$ are all located at the centers $\{x_{cm}\}$ of the sampling intervals, i.e., $\{x_m\} = \{x_{cm}\}$. Its corresponding amplitude representation is $\{r(x_{cm})\} = \{0\}$, where we assume that the Nyquist sampling criterion is satisfied for $s(x)$.

The addition of noise $n_s(x)$ to $s(x)$ prior to its sampling by the SC detector causes the interferometer output to overshoot the $(-B, B)$ range at least for some $\Delta_m$’s within $T$. The fluctuations cause the output of the (bistable) comparator to change state (with respect to its previous state at $\Delta_m$) at the partition location $x_m \neq x_m$, where $s(x_m) + n_s(x_m) = r(x_m)$. We have shown previously that, by suitable postdetection processing of $\{x_m\}$, it is possible to determine the frequency and amplitude of a weak sinusoid.11

Compared with the standard SR processing of weak sinusoids that relies on dc (fixed-threshold) crossings,1–3,8 SC sampling offers greater sensitivity. Larger $R$ values are obtained because more SC’s are obtained from $s(x) + n_s(x)$ for the same $B$ and $T$.

For our numerical experiments, we utilize uniform white noise for dithering because it yields larger $R$ values than Gaussian white noise for the same $B$ and $T$.11 Noise-aided multithreshold (equally spaced) sampling of weak sinusoids also yields an $R$ versus $\sigma$ curve that does not decrease rapidly with $\sigma$, particularly at large $N$ as compared with bithreshold ($N = 1$) detection.

At this point it is worth mentioning that if $s(x)$ is not square integrable, then it is a power signal, and it is appropriate to examine its spectral properties from its power spectrum $P_s(f) = \tilde{P}(s(x) \otimes s(x))$, where $\otimes$ represents the correlation operation.31 The discrete power spectrum $P_s(k) = \{s(m)\}$ is given by $\{P_s(k)\} = \tilde{P}(\{s(m)\} \otimes \{s(m)\})$.

In contrast, if $s(x)$ is square integrable (i.e., it is an energy signal), then its spectral properties can be analyzed from its energy spectrum $E(f) = |S(f)|^2$. The discrete energy spectrum $E_s(k) = \{S(m)\}$ is given by $E_s(k) = \{S(k)S^*(k)\}$. Random [e.g., $n_s(x)$] and periodic signals are not square integrable. However, $n_s(x)\nu(x)$ and $s(x)\nu(x)$ are square integrable as long as $T < \infty$.

We now establish all the pertinent information concerning the power spectrum $P_s(f)$ of a weak spectral doublet can be determined by noise dithering.

C. Direct Determination of $\{s(m)\}$ from the Sinusoid Crossings of $s(x) + n_s(x)$

In this subsection we show a method for determining the equally sampled representation $\{s(m)\}$ by noise dithering within $T$. We also demonstrate that the accuracy of the recovery increases with the amount of time $DT$ that is used to observe $s(x) + n_s(x) = r(x)$. The observation time is given by $DT = 2DM\Lambda$, and it is therefore necessary that $s(x)$ repeats itself $D$ times for the noise dithering technique to become accurate. Equivalently, the approach is also suitable if $s(x)$ is observed simultaneously by use of $D$ detectors and $D$ uncorrelated noise sources over one sampling period duration $T$.

We can obtain the $s(m)$ value in $\Delta_m$ by counting the number of crossings that $s(m) + n_s(m)$ makes with a chosen threshold level for a given number of $D$ trials. For $D$ independently chosen $n_s(m)$ values, the number of crossings $D_m$ that $s(m) + n_s(m)$ makes with a positive-valued threshold level $r_n(>0)$ is32

$$D_m^+ = D\{1 - \Phi_s([r_n - s(m)])\}, \quad (1)$$

where $\Phi_s(y)$ is the cumulative distribution function of the random variable $y$ and $n_s(m)$ is the value of the noise amplitude $n_s(x)$ within $\Delta_m$. For uniform white noise, $\Phi_s(y) = -y^2/2 + y^2/2\sigma$, where the probability density function $f_s(y) = (2\sigma)^{-1}$ for $|y| < \sigma$ and zero otherwise.

In contrast, the corresponding number of crossings
D_m^- that s(m) + n_s(m) makes with a negative-valued level r_n < 0 is

\[
D_m^- = D \Phi \left[ -|r_n| - s(m) \right],
\]

where \( D_{m^+} + D_{m^-} = D \). From \( D_{m^+} \) and \( D_{m^-} \), the unknown \( s(m) \) value can be calculated from the inversion of Eqs. (1) and (2):

\[
s(m) = (|r_n| - \sigma) + 2\sigma(D_{m^+}/D) \quad \text{if } D_{m^+} > D_{m^-},
\]

\[
s(m) = (\sigma - |r_n|) - 2\sigma(D_{m^-}/D) \quad \text{if } D_{m^+} < D_{m^-},
\]

\[
s(m) = 0 \quad \text{if } D_{m^+} = D_{m^-}.
\]

We obtain the complete sequence \( \{s(m)\} \) by applying Eqs. (3a)–(3c) to each of the \( 2M\Delta_m \)'s within \( T \). It should be noted that the recovery of the \( s(m) \) values by Eqs. (3a)–(3c) is realized even with only two partitions present in \( \Delta_m (N = 2) \).

The variance \( \delta s^2 \) of the calculated \( s(m) \) decreases rapidly with \( D \) according to

\[
\delta s^2 = (2\sigma/D)^2(\delta D_{m^+})^2 = (2\sigma/D)^2(\delta D_{m^-})^2,
\]

where \( (\delta D_{m^+})^2 \) is the variance of \( D_{m^+} \). Note that the variance \( \delta s^2 \) is independent of the particular \( s(m) \) value. The corresponding SNR of the measurement equals \( s(m)/\delta s = D_{m^-}/\delta D_{m^-} \) when \( |r_n| = \sigma \).

Figure 1 plots the relation between the \( D_{m^+}/D \) value and \( s(m) \) for different \( \sigma \) where \( -r_n \leq s(m) \leq r_n \). The corresponding \( D_{m^+}/D \) plots (not shown) for different \( \sigma \) values are mirror images of the \( D_{m^+}/D \) plots about \( s(m) = 0 \). For a bithreshold \( \pm r_n \) system, both the \( D_{m^+}/D \) and the \( D_{m^-}/D \) plots are used to determine the unknown \( s(m) \) value in the range \( -r_n \leq s(m) \leq r_n \).

The accuracy of the computed \( s(m) \) value depends on the high-resolution mapping between \( s(m) \) and \( D_{m^+}/D \) and \( D_{m^-}/D \) that is a function of \( \sigma \). The most accurate results are obtained by use of \( \sigma = |r_n| \). Note that it is possible to determine negative-valued \( s(m) \)’s from \( D_{m^+}/D \) alone with \( \sigma = 2r_n \) but at a reduced accuracy.

The amplitude resolution of the \( (D_{m^+}/D) \)-to-\( s(m) \) mapping improves with \( D \), which implies that the length \( DT \) of the observation time is crucial to the accurate recovery of the correct \( s(m) \) value within \( \Delta_m \). The variance \( \delta s^2 \) increases with \( \sigma \) as predicted according to (best-fit curve) \( \delta s^2 = 10^{-3}\sigma^2 \) for \( \sigma \approx r_n \). Our experiments also show that \( \delta s^2 \) is proportional to \( 1/D \) at \( \sigma = r_n \). For given values of \( s(m) \) and \( B \) and with \( \sigma = r_n \), the SNR increases at \( D^{1/2} \) because the \( D_{m^+}^- \) value is proportional with \( D \).

The main disadvantage of the above recovery procedure is that it requires long observation times (large \( D \) values) for accuracy. It is therefore difficult to apply to short-lived weak signals. In Subsection 2.D we discuss a new noise dithering procedure that requires shorter observation times to accurately determine \( \{P_s(k)\} \).

D. Algorithm to Determine the Power Spectrum of \( s(x) + n_s(x) \) by Sinusoid-Crossing Sampling

To overcome the stringent observation time requirement of the method described in Subsection 2.C, we formulate a new dithering procedure for calculating \( \{P_s(k)\} \) that requires shorter observation times for the same SNR value. It uses the fact that the \( \{P_s(k)\} \) can be calculated even without one knowing the relative phases of the various \( S(k) \) components.

From a collection of \( \{x_m\} \) that is obtained from \( D \) measurement attempts, we calculate \( \{x_m\} \) where \( \langle x_m \rangle \) represents the average SC location in \( \Delta_m \). From \( \langle x_m \rangle \), we derive the unequally sampled representation \( \{r(x_m)\} \) of \( \{s(x) + n_s(x)\} \) by \( u(x) \). For a sufficiently large \( D \), \( \{r(x_m)\} \) contains more information about \( s(x) + n_s(x) \) than \( \{r(x_m)\} \), where \( \{r(x_m)\} \) is the average \( r(x_m) \) value in \( \Delta_m \) taken over \( D \) trials.

The equally sampled representation \( \{r(m)\} = \{s(m) + n_s(m)\} \) is then interpolated from \( \{r(x_m)\} \) by use of the ideal interpolation formula\(^{15}\) that makes the interpolation procedure more efficient than the previous one\(^{11}\) that was used to determine the power spectra of weak sinusoids. Figure 2 summarizes the new dithering procedure to determine \( \{P_s(k)\} \) of \( \{r(m)\} \).

The power spectrum of \( \{r(m)\} \) is calculated as

\[
\{P_s(k)\} = \mathbb{F}[\{r(m)\} \otimes \{r(m)\}] = \mathbb{F}[\{s(m)\} \otimes \{s(m)\}] + \mathbb{F}[\{n_s(m)\} \otimes \{n_s(m)\}]
\]

\[
+ \mathbb{F}[\{n_s(m)\} \otimes \{n_s(m)\}]
\]

\[
+ \mathbb{F}[\{n_s(m)\} \otimes \{s(m)\}]
\]

\[
\approx \mathbb{F}[\{s(m)\} \otimes \{s(m)\}] + \mathbb{F}[\{n_s(m)\} \otimes \{n_s(m)\}]
\]

\[
\approx \{P_s(k)\} + \{P_s(k)\}.
\]
When rate the effects of noise from the ideal interpolation formulation. The spectrum is calculated averaged from ten independent crossing measurements. When with the FT spectrometer recently described by Daria setting also requires longer observation times. The noise dithering procedure can be implemented as the case at hand corrections of the cross terms that are not present in \( P_s(k) \). Averaging also requires longer observation times. The noise dithering procedure can be implemented with the FT spectrometer recently described by Daria and Saloma, where \( s(x) \) is the ac term of the generated interferogram and \( n_s(x) \) is the electronic noise that is added prior to detection by the 1-bit ADC. We perform the \( D \) number of measurements needed to determine \( \langle x_m \rangle \) by observing the \( x_m \) value of \( s(x) + n_s(x) \) in \( \Delta m \) for a \( D \) number of times using a new \( n_s(x) \) value each time.

### 3. Numerical Experiments

The procedure is applied to \( s(x) = (s_0/2)\exp[-\pi^2 x^2/\beta] \cos(2\pi f_1 x) + \cos(2\pi f_2 x) \) for different \( \beta = 0.5f_1 + f_2 \) and \( f_2 \), \( f_2 \geq f_2 > f_1 \). Its Fourier spectrum is \( \hat{s}(f) = (s_0/2)(\pi/\beta)^{-1/2}\exp[-\beta(f + f_1)^2] + \exp[-\beta(f + f_2)^2] + \exp[-\beta(f - f_1)^2] + \exp[-\beta(f - f_2)^2] \). The ac term \( s(x) \) represents a Gaussian-broadened doublet with resonances at \( f = \pm f_1 \) and \( \pm f_2 \), where \( S(f_1) = S(f_2) = 0.0028 \). The linewidth ratio \( L = 1.9814/[\beta^{1/2}(f_2 - f_1)] \) is set such that at \( L = 0 \) the resonances have zero linewidths and at \( L = 1 \) \( S(f) \) yields a 26.5% dip (Rayleigh resolution limit) between \( S(f_1) = S(f_2) \).

We tested whether the locations, intensities, and linewidths of \( P_s(f_1) \) and \( P_s(f_2) \) could be determined correctly using the dithering procedure that is described in Fig. 2. The following parameter values are used for \( s(x) \): \( s_0 = 0.96B, 2M = 128, N = 256, A_s = 2.5, T = 1, f_2 = 64, B = 0.01534 \), and \( f_2 - f_1 = 4 \). Because \( s_0 < B, s(x) \) is undetectable by SC sampling.

Figures 3(a)–3(c) present the recovered (discrete) power spectrum \( P_s(k) \) (solid circles) for different possible peak locations of \( f_1 = k_s/T \) and \( f_2 = k_o/T \), where \( L = 0.3, D = 100, \alpha = B, \) and \( T = 1 \). Also plotted for comparison are the corresponding correct doublet spectra \( P_s(f) \) (dotted curves). It can be seen that both the frequency locations and the linewidths of the resonances are determined accurately. However, their peak intensities are not. A relatively uniform noise background spectrum is always obtained at all frequency bands. Our experiments also revealed that the \( k_o \) and \( k_s \) positions and their associated linewidths are always determined accurately within the \( k \) range \( 3 \leq k \leq 61 \) regardless of the \( L \) value, where \( \langle k \rangle = 0.5(k_s + k_o) \).

Because \( P_s(f) \) contains several characteristic features (e.g., linewidths, peak separation, relative peak heights, line-shape profiles), the quality of its recovery is best evaluated by use of Linfoot’s criteria of fidelity \( F \), structural content \( C \), and correlation quality \( Q \). Perfect recovery yields \( F = Q = C = 1 \), where \( F = 2k - C \). Fidelity \( F \) measures the functional similarity between the recovered power spectrum \( P_s(k) \) and the correct sampled representation \( P_s(f) \) of \( P_s(f) \): \( F = 1 - \left( \langle P_s(k) - P_s(f) \rangle^2 / \langle P_s(f) \rangle^2 \right) \), where \( P_s = P_s(k), P_s = P_s(k) \), and \( \langle \cdot \rangle \) denotes an average over the entire \( k \) range. If both \( P_s(k_1) < P_s(k_1) \) and \( P_s(k_2) < P_s(k_2) \), then \( F < 1 \).

Content \( C \) measures the relative sharpness of \( P_s(k) \) with respect to \( \langle P_s(k) \rangle \): \( C = \langle P_s(k) \rangle / \langle P_s(k) \rangle \). If the resonance profiles of \( P_s(k) \) are incorrectly narrower than \( P_s(k) \), then \( C < 1 \). Quality \( Q \) measures the alignment of the peaks and troughs of \( P_s(k) \) with those of \( P_s(k) \): \( Q = \langle P_s(k) \rangle / \langle P_s(k) \rangle \). Therefore
Fig. 4. Linfoot’s criteria for recovered $|P_s(k)|$: (a) $F$ versus $\sigma$ for different $L$ values; (b) $F$, $Q$, and $C$ as a function of $L$ ($\sigma = B = 0.016$). Parameter values are $s_0 = 0.96B$, $D = 100$, $T = 1$, $2M = 128$, and $N = 25$. Dotted lines represent the true doublet spectra $|P_s(k)|$.

$L = 0.016$ ($Q = 0.83 \pm 0.08$). The appearance of spurious peaks in $P_s(k)$ prevents the $Q$ value from becoming unity. For $L = 0.4$, $F$ and $C$ are optimal at $F = 0.86 \pm 0.05$ and $C = 0.71 \pm 0.07$, respectively. The recovered peak strengths $P_s(k_1)$ and $P_s(k_2)$ are weaker than their correct values. Similar behavior for $F$, $C$, and $Q$ is also exhibited at other $k$ values in the range of $3 \leq \langle k \rangle \leq 61$. The resonances in the recovered $|P_s(k)|$ are narrower than those in $|P_s(k)|$ with the weaker side modes remaining undetermined. For $L > 0.4$, $C$ increases away from unity, implying the calculation of incorrect values for the resonance profiles.

Figure 5(a) plots the percentage error $\epsilon_1$ versus $D$ for $\sigma = B$, where $\epsilon_1 = 100 \left(\langle P_s(k_1) \rangle - P_s(k_1) \right)$, and $\langle P_s(k_1) \rangle$ equals the average of $P_s(k_1)$ over $(D/10)$ trials ($100 \leq D \leq 5000$). The calculation involves $D/10 P_s(k_1)$’s that are randomly selected from a set of 600 values in the range of $3 \leq \langle k \rangle \leq 61$. Percentage errors are also calculated for $P_s(k_2)$. The errors obtained are $\epsilon_1 = 15.6\% \pm 2.4$ and $\epsilon_2 = 19.7\% \pm 2.8$ for $L = 0$, and $\epsilon_1 = 40.1\% \pm 0.4$ and $\epsilon_2 = 37.3\% \pm 0.4$ for $L = 0.4$. Accurate recovery becomes more difficult with increasing widths of the resonance lines.

Figure 5(b) plots the $SNR_1$ and $SNR_2$ values as a
function of $D$ for $L = 0$ and $0.4$. At $L = 0$ and $s_0 = 0.96B$, SNR$_1$ equals 49.1 ± 1.2 and SNR$_2$ equals 50.3 ± 1.1. For doublets with broader spectral profiles ($L = 0.4$), we obtained a SNR$_1$ equaling 10.4 ± 0.6 and a SNR$_2$ equaling 10.9 ± 0.7. For $L = 0.4$, the values of $\epsilon_1$, $\epsilon_2$, SNR$_{1s}$, and SNR$_{2s}$ do not vary significantly with $D$. In the absence of noise [$s_f(x) = 0$], no information is obtained about $P_s(f)$ and therefore SNR equals 0.

Figure 5(c) presents the dependence of SNR$_1$ (solid circles) and SNR$_2$ (open circles) with $s_0$ that determines the peak value of $s(x)$.

Dithering is done with $\sigma = B$ and for $D = 100$. Results show that both SNR$_1$ and SNR$_2$ increase linearly with $s_0$. At $L = 0$, the increase with $s_0$ is described by (best-fit curve) a SNR$_1$ equaling $58.98s_0^{1.03}$ and a SNR$_2$ equaling $63.1s_0^{1.09}$, where $0.01B \leq s_0 \leq 0.99B$. At $s_0 = 0.13B$, SNR$_1$ is approximately 11 and SNR$_2$ is approximately 8; at $s_0 = 0.99B$, SNR$_1$ is approximately 56.5 and SNR$_2$ is approximately 45.2.

4. Discussion

Our numerical experiments have shown that the dithering procedure described in Subsection 2.D could determine the locations of the doublet resonances with 100% accuracy in the average frequency range of $3 \leq (k) \leq 61$, even for broadly profiled resonances ($L = 1$). The dithering procedure works best when the uniform white-noise variance is such that $\sigma = B$ [see Fig. 4(a)]. For $L < 0.4$ and for $\sigma = 0.016 = B$ [see Fig. 4(b)], the three Linfoot’s criteria

Fig. 5. Effect of spectral averaging in the recovery of unknown $P_s(k_1)$ and $P_s(k_2)$ values for $L = 0$ and $L = 0.4$: (a) Percentage error versus $D$ showing $\epsilon_1(L = 0$, solid circles), $\epsilon_2(L = 0$, open circles), $\epsilon_3(L = 0.4$, crosses), and $\epsilon_4(L = 0.4$, open squares); (b) SNR versus $D$ showing SNR$_1$ (0, solid circles), SNR$_2$ (0, open circles), SNR$_{N1}$ (0.4, crosses), and SNR$_{N2}$ (0.4, open squares); and (c) SNR, and SNR$_s$ versus $s_5$ for $D = 100$, showing SNR$_1$ (0, solid circles), SNR$_2$ (0, open circles), SNR$_{N1}$ (0.4, crosses), SNR$_{N2}$ (0.4, open squares). Parameter values are $\sigma = B = 0.016$, $T = 1$, $2M = 128$, and $N = 256$. The $\langle P_s(k) \rangle$ and $\langle x_m \rangle$ are calculated from $D/10$ and ten trials, respectively.

have values that are close to unity, indicating a good recovery not only of the resonance locations but also of the resonance profiles.

The recovery procedure achieves significant gain in the SNR at a shorter observation time $DT$. With each $\langle P_s(k) \rangle$ calculated by ten trials for $\langle x_m \rangle$, a SNR value of approximately 50 [see Fig. 5(b)] is gained for both $\langle P_s(k_1) \rangle$ and $\langle P_s(k_2) \rangle$ for $L = 0$ with the spectral components averaged over $D/10$ independently calculated $\langle P_s(k) \rangle$’s. For a given $D$, the $P_s(f)$’s with narrower linewidths are easier to recover; SNR values of approximately 50 and 10 are gained for $L = 0$ and $L = 0.4$, respectively. For sufficiently large $D$ values, the maximum SNR that is attainable depends primarily on the threshold $B$ value [see Fig. 5(c)].

Figure 5(b) also shows that the SNR does not increase further with increasing $D$, which implies that averaging over a large number of $\{P_s(k)\}$’s does not improve the performance of the recovery with the ten trials that are always used for $\langle x_m \rangle$. No information about $\langle P_s(k) \rangle$ is obtained from $\langle P_s(k) \rangle$ is given by the average of the power spectra that is derived directly from the individual $\{x_m\}$’s without averaging. This characteristic illustrates why it is important to first average over the $x_m$’s before $\langle P_s(k) \rangle$ is calculated.

To illustrate this point, in Fig. 6 we present how the SNR increases with the number of trials $D/10$ used to average each $\langle x_m \rangle$. In all cases the average power spectrum $\langle P_s(k) \rangle$ is taken from ten trials. The results show that $P_s(f)$’s with narrower linewidths are again easier to recover.

For $L = 0$, SNR$_2$ (open circles) approximately increases in a logarithmic manner with $D/10$. The knee in the best-fit logarithmic curve occurs near $D/10 = 250$. An examination of the recovered power spectra reveals that when a large number of trials are used to calculate $\langle x_m \rangle$, the average background noise $\langle N \rangle$ is reduced in $\langle P_s(k) \rangle$. It can be seen that the quality of the recovery is better in
\[ \langle P_s(k_s) \rangle \] than \( \langle P_o(k_s) \rangle \) with an increasing \( D/10 \) value.

The SNR value increases linearly with \( s_0 \). The \( \{P_s(k_s)\}'s \) of spectral doublets with weak ac terms of the interferogram are harder to recover than those with peak values that are close to the detection limit \( B \) [see Fig. 5(c)].

5. Conclusion

We have proposed and tested an efficient noise dithering procedure for accurately measuring the power spectra of weak doublet resonances by a FT spectrometer. The subthreshold ac term of the interferogram is measured by SC sampling with a 1-bit ADC. Without the addition of noise, the ac term is undetectable by the ADC. The dithering procedure achieves optimal gains in the SNR's of the recovered power spectra at shorter observation times than the dithering procedure that directly recovers \( \{s(m)\} \) where the variance \( \delta s^2 \) of the recovered \( s(m) \) value is directly proportional to \( 1/D^2 \).

Appendix A: Parameters Used in This Paper

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s(x) )</td>
<td>ac term of interferogram</td>
</tr>
<tr>
<td>( x )</td>
<td>Optical path difference of two beams in a Michelson interferometer</td>
</tr>
<tr>
<td>( \otimes )</td>
<td>Correlation operation</td>
</tr>
<tr>
<td>( \mathbb{F} )</td>
<td>Fourier-transform operator</td>
</tr>
<tr>
<td>( S(f) = \mathbb{F}[s(x)] )</td>
<td>Fourier spectrum of ( s(x) )</td>
</tr>
<tr>
<td>( P_s(f) )</td>
<td>Power spectrum of ( s(x) )</td>
</tr>
<tr>
<td>( n_s(x) )</td>
<td>Additive noise of variance ( \sigma^2 )</td>
</tr>
<tr>
<td>( s(x) + n_s(x) )</td>
<td>Dithered signal</td>
</tr>
<tr>
<td>( P_o(f) )</td>
<td>Power spectrum of ( s(x) + n_s(x) )</td>
</tr>
<tr>
<td>( \text{SNR} )</td>
<td>Signal-to-noise ratio</td>
</tr>
<tr>
<td>( r(x) = A_s \cos(2\pi f_s x) )</td>
<td>Reference sinusoid of SC sampling</td>
</tr>
<tr>
<td>( {x_m} )</td>
<td>SC locations of ( s(x) ) with ( r(x) )</td>
</tr>
<tr>
<td>( \Delta_m = 1/2f_s )</td>
<td>( m )th-sampling interval, where ( m = 1, 2, \ldots, 2M )</td>
</tr>
<tr>
<td>( 2M )</td>
<td>Numbers of ( \Delta_m )'s in one sampling period ( T )</td>
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<tr>
<td>( \delta = \Delta_m/N )</td>
<td>Partition size within ( \Delta_m )</td>
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<tr>
<td>( x_{cm} )</td>
<td>Center position of ( \Delta_m )</td>
</tr>
<tr>
<td>( r_0 = B = \pi A_s/2N )</td>
<td>Threshold level of SC sampling</td>
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<tr>
<td>( \Delta r_n )</td>
<td>Amplitude separation between ( n ) and ( n - 1 ) levels</td>
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<tr>
<td>( {r(m)} )</td>
<td>Equally sampled values of dithered signal</td>
</tr>
<tr>
<td>( D_m^+ )</td>
<td>Number of crossings of ( {s(x) + n_s(x)} ) with ( r_n(&gt;0) )</td>
</tr>
<tr>
<td>( D_m^- )</td>
<td>Number of crossings of ( {s(x) + n_s(x)} ) with ( r_n(&lt;0) )</td>
</tr>
<tr>
<td>( \Phi_y(y) )</td>
<td>Cumulative distribution function of random variable ( y )</td>
</tr>
<tr>
<td>( \delta s^2 )</td>
<td>Variance of calculated ( s(m) ) value</td>
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<tr>
<td>( (\delta D_m^2) )</td>
<td>Variance of ( D_m )</td>
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<td>( L )</td>
<td>Linfoot’s correlation quality criterion</td>
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<td>( {P_s(k_s)} )</td>
<td>Sampled representation of ( P_s(f) )</td>
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<td>( {S(k_s)} )</td>
<td>Sampled representation of ( S(f) )</td>
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<tr>
<td>( {P_o(k_s)} )</td>
<td>Power spectrum of ( {r(m)} )</td>
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<td>( e )</td>
<td>Percentage error</td>
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<td>( w(x) )</td>
<td>Sampling window function</td>
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References

19. C. Zierhofer, “Adaptive sigma-delta modulation with one-bit